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Generalized intelligent states for an arbitrary quantum system

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Abstract

Generalized intelligent states (coherent and squeezed states) are derived for an arbitrary quantum system by using the minimization of the so-called Robertson–Schrödinger uncertainty relation. The Fock–Bargmann representation is also considered. As a direct illustration of our construction, the Pöschl–Teller potentials of trigonometric type will be chosen. We will show the advantage of the Fock–Bargmann representation in obtaining the generalized intelligent states in an analytical way. Many properties of these states are studied.

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1. Introduction

The well known coherent states of the harmonic oscillator have turned out to be one of the most useful tools of quantum theory [1–4]. Introduced long ago by Schrödinger [5], they were employed by Glauber and other authors in quantum optics [6–8]. Further developments of the subject made it possible to set up some specific definitions, which are applicable to various physical systems. They were discussed in connection with exactly solvable models and nonlinear algebras [9–12] as well as deformed algebras [13].

Recently, a construction of coherent states for an arbitrary quantum system has been proposed by Gazeau and Klauder [14] (see also [15] and [16]). An interesting illustration of this construction was given in [17] for a particle trapped in an infinite square well and in Pöschl–Teller potentials of trigonometric type [18]. The Gazeau–Klauder coherent states [14] are eigenvectors of the annihilation operator.

On the other hand, the squeezed states of an electromagnetic field have attracted due attention in the last decade (see for instance [3, 18]). In recent years, considerable interest has also been devoted to the squeezed states for spin components [19, 20], the number and phase operators [21], the generators of the algebras $su(2)$ and $su(1,1)$ [22–24] and the supersymmetric oscillator [25].

The aim of this paper is to consider some general properties of generalized intelligent states for an arbitrary quantum system. These states minimize [26] the Robertson–Schrödinger uncertainty relation [27, 28] and generalize the Gazeau–Klauder coherent states [14]. Coherence and squeezing are discussed throughout this paper.

The paper is organized as follows. In section 2, we recall the main results concerning the states minimizing the Robertson–Schrödinger uncertainty relation and some useful formulae which are relevant in the study of coherence and squeezing of these states. The generalized intelligent states minimizing the Robertson–Schrödinger uncertainty relation are explicitly computed in section 3 and we show that they generalize the Gazeau–Klauder coherent states in some special cases, which will be discussed. In section 4, we introduce the Fock–Bargmann realization of the Gazeau–Klauder coherent states by means of which we construct, in section 5, the Pöschl–Teller intelligent states. Coherence and squeezing of such states are also considered. Conclusions and concluding remarks are given in section 6.

2. Robertson–Schrödinger uncertainty relation

Choose a Hamiltonian H with a discrete spectrum which is bounded below, and has been adjusted so that $H \geqslant 0$. For convenience, we assume that the eigenstates of H are non-degenerate. The eigenstates $|\psi_n\rangle$ of H are orthonormal vectors and they satisfy

$$H\,|\psi_n\rangle = e_n\,|\psi_n\rangle\,. \tag{1}$$

In a general setting, we also assume that the energies $e_0, e_1, e_2, \ldots$ are positive and verify $e_{n+1} > e_n$. The ground-state energy $e_0 = 0$. Therefore, there is a dynamical algebra generated by lowering and raising operators a^+(creation operator) and a^-(annihilation operator) such that the Hamiltonian H can be factorized as

$$H = a^+a^-. \tag{2}$$

The actions of the operators a^+ and a^-on $|\psi_n\rangle$ are given by

$$\begin{aligned} a^-\,|\psi_n\rangle &= \sqrt{e_n}\mathrm{e}^{\mathrm{i}\alpha(e_n-e_{n-1})}\,|\psi_{n-1}\rangle \\ a^+\,|\psi_n\rangle &= \sqrt{e_{n+1}}\mathrm{e}^{-\mathrm{i}\alpha(e_{n+1}-e_n)}\,|\psi_{n+1}\rangle \qquad \alpha\in\mathbb{R} \end{aligned} \tag{3}$$

implemented by the action of a^- on the ground state $|\psi_0\rangle$

$$a^-\,|\psi_0\rangle = 0. \tag{4}$$

The exponential factor appearing in all expressions produces only a phase factor, and will be significant for the temporal stability of the generalized intelligent states, which we will construct in the following.

The commutator of a^+ and a^- is defined by

$$\left[a^-, a^+\right] = G\big(N\big) \equiv G \tag{5}$$

where the operator $G\,(N)$ is defined by its action on states $|\psi_n\rangle$

$$G(N)\big|\psi_n\big\rangle = \big(e_{n+1} - e_n\big)\big|\psi_n\big\rangle. \tag{6}$$

It is diagonal with eigenvalues $(e_{n+1} - e_n)$. We define the operator number N as

$$N\big|\psi_n\big\rangle = n\big|\psi_n\big\rangle \tag{7}$$

which is in general different from the product $a^+a^-(= H)$. We can see that it satisfies the following commutation relations:

$$\left[a^-, N\right] = a^- \qquad \left[a^+, N\right] = -a^+. \tag{8}$$

Using a^+ and a^-, we introduce two Hermitian operators

$$W = \frac{1}{\sqrt{2}}\left(a^- + a^+\right) \qquad P = \frac{\mathrm{i}}{\sqrt{2}}\left(a^+ - a^-\right) \tag{9}$$

which satisfy the commutation relation

$$\left[W, P\right] = \mathrm{i}G. \tag{10}$$

The operator G, in general, is not necessarily a multiple of the unit operator. It is well known that for two Hermitian operators W and P satisfying the non-canonical commutation relation (10), the variances $(\Delta W)^2$ and $(\Delta P)^2$ satisfy the Robertson–Schrödinger uncertainty relation

$$\left(\Delta W\right)^2\left(\Delta P\right)^2 \geqslant \tfrac{1}{4}\left(\langle G\rangle^2 + \langle F\rangle^2\right) \tag{11}$$

where the operator F is defined by

$$F = \{W - \langle W\rangle, P - \langle P\rangle\} \tag{12}$$

or by

$$F = \mathrm{i}\left[\left(2a^- - \langle a^-\rangle\right)\langle a^-\rangle + \left(-2a^+ + \langle a^+\rangle\right)\langle a^+\rangle - a^{-2} + a^{+2}\right] \tag{13}$$

in terms of the operators a^- and a^+.

The symbol $\{,\}$ in (12) stands for the anti-commutator. When there is a correlation between W and P, i.e. $\langle F\rangle \neq 0$, the relation (11) is a generalization of the usual one (the Heisenberg uncertainty condition)

$$(\Delta W)^2\,(\Delta P)^2 \geqslant \tfrac{1}{4}\langle G\rangle^2. \tag{14}$$

The special form (14) is, of course, identical with the general form (11) if W and P are uncorrelated, i.e. if $\langle F\rangle = 0$. The general uncertainty relation (11) is better suited to determine the lower bound on the product of variances in the measurement of observables corresponding to the non-canonical operators. The Robertson–Schrödinger uncertainty relation gives us a new understanding of which states are coherent and which are squeezed for an arbitrary quantum system. Indeed the so-called generalized intelligent states are obtained when the equality in the Robertson–Schrödinger relation is realized [26]. The inequality in (11) becomes the equality for the states $|\Psi\rangle$ satisfying the equation

$$(W + \mathrm{i}\lambda P)\,|\Psi\rangle = z\sqrt{2}\,|\Psi\rangle \qquad \lambda, z \in \mathbb{C}. \tag{15}$$

As a consequence, we have the following relations:

$$(\Delta W)^2 = |\lambda|\,\Delta \qquad (\Delta P)^2 = \frac{1}{|\lambda|}\Delta \tag{16}$$

with

$$\Delta = \tfrac{1}{2}\sqrt{\langle G\rangle^2 + \langle F\rangle^2}. \tag{17}$$

Note that the average values $\langle G\rangle$ and $\langle F\rangle$, in the states satisfying the eigenvalue equation (15), can be expressed in terms of the variances as

$$\begin{aligned}\langle G\rangle &= 2\mathrm{Re}\;(\lambda)\,(\Delta P)^2\\ \langle F\rangle &= 2\mathrm{Im}\;(\lambda)\,(\Delta P)^2\,.\end{aligned} \tag{18}$$

It is clear, from (16), that if $|\lambda| = 1$ we have

$$(\Delta W)^2 = (\Delta P)^2 \tag{19}$$

and we call the states satisfying (15) with $|\lambda| = 1$ the generalized coherent states; if $|\lambda| \neq 1$ the states are called generalized squeezed states.

Using the equation (15), one can obtain some general relations for the average values and dispersions for W and P in the states which minimize the Robertson–Schrödinger uncertainty relation (11). Indeed, we have

$$(\Delta W)^2 = \frac{1}{2}\left(\mathrm{Re}\ (\lambda)\ \langle G\rangle + \mathrm{Im}\ (\lambda)\ \langle F\rangle\right) \tag{20}$$

$$(\Delta P)^2 = \frac{1}{2\,|\lambda|^2}\left(\mathrm{Re}\ (\lambda)\ \langle G\rangle + \mathrm{Im}\ (\lambda)\ \langle F\rangle\right) \tag{21}$$

$$\mathrm{Im}\ (\lambda)\ \langle G\rangle = \mathrm{Re}\ (\lambda)\ \langle F\rangle . \tag{22}$$

We conclude this section by noticing that the minimization of the Robertson–Schrödinger uncertainty relation leads to generalized coherent states for $|\lambda| = 1$ (including the so-called Gazeau–Klauder states obtained here for $\lambda = 1$, which minimize the Heisenberg uncertainty condition and are eigenvectors of the annihilation operator a^-), and generalized squeezed states for $|\lambda| \neq 1$.

3. Generalized intelligent states

In the following, we will solve the eigenvalue equation (15) in order to give a complete classification of the coherent and squeezed states for an arbitrary quantum system. To solve the eigenvalue equation (15), it is convenient to use the definition of W and P in terms of the creation and annihilation operators a^+ and a^-. The equation (15) is rewritten in the following form:

$$\left\{(1-\lambda)\, a^+ + (1+\lambda)\, a^-\right\} |\Psi\rangle = 2z\, |\Psi\rangle . \tag{23}$$

Let us compute $|\Psi\rangle$ explicitly using (23). We take

$$|\Psi\rangle = \sum_{n=0}^{\infty} c_n\, |\psi_n\rangle \tag{24}$$

so that

$$\begin{aligned} &(1-\lambda)\, c_{n-1}\sqrt{e_n}\mathrm{e}^{-\mathrm{i}\alpha(e_n-e_{n-1})} + (1+\lambda)\, c_{n+1}\sqrt{e_{n+1}}\mathrm{e}^{\mathrm{i}\alpha(e_{n+1}-e_n)} = 2zc_n \\ &(1+\lambda)\sqrt{e_1}c_1 = 2z\mathrm{e}^{-\mathrm{i}\alpha e_1}c_0. \end{aligned} \tag{25}$$

Using the latter relation, let us give a complete classification of generalized intelligent states for an arbitrary quantum system. We will analyse the solution for the following cases: ($\lambda = 1$, $z \neq 0$), ($\lambda = -1$, $z \neq 0$), ($\lambda \neq -1$, $z = 0$) and ($\lambda \neq -1$, $z \neq 0$). In each case, we give the solution of the equation (23) as some operator acting on the ground state $|\psi_0\rangle$ of the quantum system under consideration.

3.1. Gazeau–Klauder coherent states

As we will see, this set of states correspond to the situation where ($\lambda = 1$, $z \neq 0$). In this case, the equations (25) above are rewritten as

$$c_n = z^n \frac{\mathrm{e}^{-\mathrm{i}\alpha e_n}}{\sqrt{e_n e_{n-1}\ldots e_1}} c_0.$$

Then, the coherent states are given by

$$|\Psi\rangle = |z,\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{z^n \mathrm{e}^{-\mathrm{i}\alpha e_n}}{\sqrt{f\ (n)}}\, |\psi_n\rangle \tag{26}$$

where the function $f(n)$ is defined by

$$f(n) = \begin{cases} e_n e_{n-1} \ldots e_1 & \text{for} \quad n \neq 0 \\ 1 & \text{for} \quad n = 0. \end{cases} \tag{27}$$

The normalization constant c_0 is calculated from the normalization condition

$$\langle z, \alpha \, | z, \alpha\rangle = 1 \tag{28}$$

and is given by

$$c_0 = \left(\sum_{n=0}^{\infty} \frac{|z|^{2n}}{f(n)} \right)^{-\frac{1}{2}} \tag{29}$$

the coherent states obtained here are, then, solutions of the eigenvalue equation

$$a^- |\Psi\rangle = z |\Psi\rangle \,. \tag{30}$$

In other words, the states $|\Psi\rangle$ are the eigenstates of the annihilation operator. This is one of the possibilities to define the coherent states. It is well known that there are several non-equivalent definitions of them for a general system [1, 2]. In the arbitrary quantum system case, the connection with a possible group-theoretical approach cannot be applied because, in contrast to the harmonic oscillator, the operators a^+, a^- and G do not close a Lie algebra. If we want to obtain the usual Heisenberg–Weyl algebra, we have to modify these operators for new ones, which will be labelled by A^- and A^+, and satisfy the canonical commutation relation

$$\left[A^-, A^+\right] = 1. \tag{31}$$

Let us take

$$A^- = a^- \qquad A^+ = \frac{N}{g(N)} a^+ \tag{32}$$

where the operator $g(N)$ is defined by

$$g(N) = a^+ a^- = H. \tag{33}$$

Note that A^- and A^+ are not self-adjoint. So, it is possible to rewrite $|\Psi\rangle$ (up to normalization) as

$$|\Psi\rangle = \exp\left(z \frac{N}{g(N)} a^+ \right) |\psi_0\rangle \,. \tag{34}$$

We see then that the coherent states ($\lambda = 1$) minimize the Heisenberg uncertainty relation, and are defined as eigenvectors of the annihilation operator a^-. They can also be given as the action of operator $\exp(zA^+)$ on the ground state $|\psi_0\rangle$.

It is easy to see that for $\lambda = 1$ we have

$$(\Delta W)^2 = (\Delta P)^2 = \tfrac{1}{2} \langle G\rangle \tag{35}$$

where

$$\langle G\rangle = c_0^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{f(n)} e_{n+1} - |z|^2 \tag{36}$$

and

$$\langle F\rangle = 0. \tag{37}$$

The latter equation traduces the fact that there is no correlation between W and P.

Let us note that our coherent states coincide with those proposed by Gazeau and Klauder [14], where a set of four requirements of such states has been imposed, i.e. continuity, resolution of unity, temporal stability and action identity. Let us now verify that our coherent states satisfy all these requirements. However, it should be noted that our coherent states satisfy additional properties. They minimize the Heisenberg uncertainty condition, are eigenstates of a^- and are given as the action of the operator $\exp(zA^+)$ on the ground state $|\psi_0\rangle$ (see equation (34)).

We see that they are continuous in $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. Moreover, the presence of the phase factor in the definition (3) of the action of a^- and a^+ leads to temporal stability of the coherent states. Indeed, we have

$$\mathrm{e}^{\mathrm{i}Ht}\,|z,\alpha\rangle = |z,\alpha+t\rangle\,. \tag{38}$$

The analysis of completeness (in fact, the overcompleteness) requires us to compute the resolution of the identity, that is

$$\int |z,\alpha\rangle\,\langle z,\alpha|\;\mathrm{d}\mu\,(z) = I_{\mathcal{H}}. \tag{39}$$

Note that the integral is over the disc $\{z \in \mathbb{C}, |z| < \mathcal{R}\}$, where the radius of convergence $\mathcal{R}$ is

$$\mathcal{R} = \lim_{n\to\infty} \sqrt[n]{f(n)} \tag{40}$$

and the measure $\mathrm{d}\mu\,(z)$ has to be determined. We suppose that $\mathrm{d}\mu\,(z)$ depends only on $|z|$. Then taking

$$\mathrm{d}\mu\,(z) = [c_0]^2\,h(r^2)r\,\mathrm{d}r\,\mathrm{d}\phi \qquad z = r\mathrm{e}^{\mathrm{i}\phi} \tag{41}$$

and using the coherent states (given by formula (26)), we can write (39) as

$$I_{\mathcal{H}} = \sum_{n=0}^{\infty} |\psi_n\rangle\,\langle\psi_n| \left[\frac{\pi}{f(n)} \int_0^{\mathcal{R}^2} h(u)u^n\,\mathrm{d}u\right]. \tag{42}$$

The resolution of the identity is then equivalent to the determination of the function $h(u)$ satisfying

$$\int_0^{\mathcal{R}^2} h(u)u^{n-1}\,\mathrm{d}u = \frac{f(n-1)}{\pi}. \tag{43}$$

For $\mathcal{R} \to \infty$, it is clear that $h(u)$ is the inverse Mellin transform of $\frac{f(n-1)}{\pi}$

$$h(u) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{f(s-1)}{\pi} u^{-s}\,\mathrm{d}s \qquad c \in \mathbb{R}. \tag{44}$$

Note that explicit calculation of the function $h(u)$ requires the explicit knowledge of the spectrum of the quantum system under consideration. The measure of the coherent states is related to the spectrum and the special application was treated for the Pöschl–Teller potential [17], Morse potential [29] and Jaynes–Cummings model [30].

Using equation (30), one can obtain the mean value of the Hamiltonian H in the states $|z,\alpha\rangle$

$$\langle z,\alpha|\,H\,|z,\alpha\rangle = |z|^2\,. \tag{45}$$

This relation is known as the action identity. It is clear now that our coherent states also satisfy the Gazeau–Klauder requirements absolutely necessary to define coherent states for an arbitrary quantum system.

3.2. The case ($\lambda = -1, z \neq 0$)

In this case, we have to solve the eigenvalue equation

$$a^{+} |\Psi\rangle = z |\Psi\rangle . \tag{46}$$

Then, the recurrence relations (25) are rewritten

$$z c_n = c_{n-1} \sqrt{e_n} \mathrm{e}^{-\mathrm{i}\alpha(e_n - e_{n-1})} \qquad \text{and} \qquad c_0 = 0. \tag{47}$$

Then all coefficients vanish and we conclude that the solution in this case cannot be normalized. The case $\lambda = -1$, leading to the unnormalized solution, is not of interest.

3.3. The case ($\lambda \neq -1, z = 0$)

In the case where $\lambda \neq -1$, we will produce completely the set of solutions which will give squeezed ($|\lambda| \neq 1$) and generalized coherent states ($|\lambda| = 1$). We start by examining the special case where $\lambda \neq -1$ and $z = 0$ in order to have an idea about the general solution of the eigenvalue equation in the general case corresponding to the situation where $\lambda \neq -1$ and $z \neq 0$. Therefore, in the case $\lambda \neq -1$, $z = 0$, expanding the state $|\Psi\rangle$ as

$$|\Psi\rangle = \sum_{n=0}^{\infty} c_n |\psi_n\rangle \tag{48}$$

and using equations (25), one can see that the coefficients c_n satisfy the following recurrence formulae:

$$(1+\lambda)\, c_{n+1} \sqrt{e_{n+1}} \mathrm{e}^{\mathrm{i}\alpha(e_{n+1} - e_n)} = (\lambda - 1)\, c_{n-1} \sqrt{e_n} \mathrm{e}^{-\mathrm{i}\alpha(e_n - e_{n-1})} \tag{49}$$

and

$$c_1 = 0. \tag{50}$$

Then, the solution of equation (23) is a linear combination of the states $|\psi_{2k}\rangle$ ($k = 0, 1, 2, 3 \ldots$)

$$|\Psi\rangle = \sum_{k=0}^{\infty} c_{2k} |\psi_{2k}\rangle \tag{51}$$

where the coefficients c_{2k} are given by

$$c_{2k} = \left(\frac{\lambda - 1}{\lambda + 1}\right)^{k} \sqrt{\frac{e_1 e_3 \ldots e_{2k-1}}{e_2 e_4 \ldots e_{2k}}} \mathrm{e}^{-\mathrm{i}\alpha e_{2k}} c_0 \qquad k \geqslant 1. \tag{52}$$

(Note that the coefficients $c_{2k-1} = 0$ for $k \geqslant 1$.) The coefficients c_0 can be calculated by imposing the normalization condition: $\langle\Psi\, |\Psi\rangle = 1$. We obtain

$$c_0 = \left[\sum_{k=1}^{\infty} \left|\frac{\lambda - 1}{\lambda + 1}\right|^{2k} \frac{(e_1 e_3 \ldots e_{2k-1})^2}{f(2k)}\right]^{-\frac{1}{2}} \qquad k \geqslant 1. \tag{53}$$

It is interesting to mention that the state $|\Psi\rangle$ can be obtained by the action of the operator

$$U(\lambda \neq -1, z = 0) = c_0 \exp\left(\frac{1}{2}\left(\frac{\lambda - 1}{\lambda + 1}\right) \frac{N}{g(N)} \left(a^{+}\right)^2\right) \tag{54}$$

on the ground state $|\psi_0\rangle$, where $g(N)$ is defined as in (33). So, the states minimizing the Robertson–Schrödinger uncertainty relation with $\lambda \neq -1$ and $z = 0$ are given by

$$|\Psi\rangle = U(\lambda \neq -1, z = 0) |\psi_0\rangle . \tag{55}$$

Note that for $\lambda = 1$ we have

$$U(\lambda = 1, z = 0) = c_0 \tag{56}$$

and the states $|\Psi\rangle$ are nothing but the ground state $|\psi_0\rangle$, which is annihilated by the operator a^- ($a^- |\psi_0\rangle = 0$).

The result of this subsection can be seen as a first step to obtain the generalized intelligent states for an arbitrary quantum system.

3.4. The case ($\lambda \neq -1, z \neq 0$)

This case is more interesting and leads to generalized intelligent states for an arbitrary quantum system. We start by solving the eigenvalue equation (23) and we give the solution of this equation as the action of some operator, which will be defined later on, on the ground state of the system under consideration. The example of the harmonic oscillator is discussed at the end of this section.

In the case where $\lambda \neq -1$ and $z \neq 0$, the eigenvalue equation (23) gives the following recurrence formulae:

$$(1-\lambda)\, c_{n-1}\sqrt{e_n}\mathrm{e}^{-\mathrm{i}\alpha(e_n-e_{n-1})} + (1+\lambda)\, c_{n+1}\sqrt{e_{n+1}}\mathrm{e}^{\mathrm{i}\alpha(e_{n+1}-e_n)} = 2zc_n \tag{57}$$

and

$$c_1 = \frac{2z\mathrm{e}^{-\mathrm{i}\alpha e_1}}{(1+\lambda)\sqrt{e_1}} c_0. \tag{58}$$

Setting

$$A_{n+1} = \frac{c_{n+1}}{c_n}\sqrt{e_{n+1}}\mathrm{e}^{\mathrm{i}\alpha(e_{n+1}-e_n)} \tag{59}$$

the relations (58) and (59) can be written in the following form:

$$A_1 = \frac{2z}{(1+\lambda)} \qquad \text{and} \qquad A_n = \frac{2z}{(1+\lambda)} + \left(\frac{\lambda-1}{\lambda+1}\right)\frac{e_{n-1}}{A_{n-1}}. \tag{60}$$

From the latter equations, we obtain the coefficients A_n, which are expressed as continued fractions. They are given by

$$A_n = \frac{2z}{1+\lambda} + \cfrac{\left(\frac{\lambda-1}{\lambda+1}\right)e_{n-1}}{\frac{2z}{1+\lambda} + \cfrac{\left(\frac{\lambda-1}{\lambda+1}\right)e_{n-2}}{\frac{2z}{1+\lambda} + \cfrac{\left(\frac{\lambda-1}{\lambda+1}\right)e_{n-3}}{\frac{2z}{1+\lambda} + \cdots \ddots \frac{2z}{1+\lambda} + \frac{\left(\frac{\lambda-1}{\lambda+1}\right)e_1}{A_1}}}}. \tag{61}$$

Now we are able to compute the coefficients c_n. Indeed, they are given by the following expression:

$$c_n = c_0 \frac{(2z)^n}{(1+\lambda)^n\sqrt{f(n)}}\left[\sum_{h=0(1)\left[\frac{n}{2}\right]}(-1)^h \frac{\left(1-\lambda^2\right)^h}{(2z)^{2h}}\Delta(n,h)\right]\mathrm{e}^{-\mathrm{i}\alpha e_n} \tag{62}$$

where the symbol $\left[\frac{n}{2}\right]$ represents the integer part of $\frac{n}{2}$ and the function $\Delta(n,h)$ is defined by

$$\Delta(n,h) = \sum_{j_1=1}^{n-(2h-1)} e_{j_1}\left[\sum_{j_2=j_1+2}^{n-(2h-3)} e_{j_2}\ldots\left[\ldots\left[\sum_{j_h=j_{h-1}+2}^{n-1} e_{j_h}\right]\right]\ldots\right]. \tag{63}$$

As an example of computation of the c_n, we give the first four coefficients

$$c_1 = \frac{2z}{(1+\lambda)\sqrt{e_1}} \mathrm{e}^{-\mathrm{i}\alpha e_1} c_0$$

$$c_2 = \frac{(2z)^2}{(1+\lambda)^2\sqrt{e_1 e_2}} \left[1 + \frac{\lambda^2-1}{(2z)^2} e_1\right] \mathrm{e}^{-\mathrm{i}\alpha e_2} c_0$$

$$c_3 = \frac{(2z)^3}{(1+\lambda)^3\sqrt{e_1 e_2 e_3}} \left[1 + \frac{\lambda^2-1}{(2z)^2} (e_1 + e_2)\right] \mathrm{e}^{-\mathrm{i}\alpha e_3} c_0$$

$$c_4 = \frac{(2z)^4}{(1+\lambda)^4\sqrt{e_1 e_2 e_3 e_4}} \left[1 + \frac{\lambda^2-1}{(2z)^2} (e_1 + e_2 + e_3) + \left(\frac{\lambda^2-1}{(2z)^2}\right)^2 e_1 e_3\right] \mathrm{e}^{-\mathrm{i}\alpha e_4} c_0.$$

As we mentioned at the beginning of this section, the general solution of the eigenvalue equation (23) can be written as the action of some operator on the ground state $|\psi_0\rangle$ of H. A more or less complicated calculation would give the following result:

$$|\Psi\rangle = U\,(\lambda \neq -1, z \neq 0)\,|\psi_0\rangle \tag{64}$$

where

$$U\,(\lambda \neq -1, z \neq 0) = c_0 \sum_{n=0}^{\infty} \left(\left(\frac{2z}{\lambda+1}\right) \frac{a^+}{g\,(N)} + \left(\frac{\lambda-1}{\lambda+1}\right) \frac{1}{g\,(N)} \left(a^+\right)^2 \right)^n. \tag{65}$$

In the case where $\lambda = 1$, we recover the operator which, acting on $|\psi_0\rangle$, gives the Gazeau–Klauder coherent states.

Note also that for $\lambda \neq -1$ and $z = 0$, the operator (65) coincides with that given by (54). Moreover, it is not difficult to see that the generalized intelligent states are stable temporally. Finally, as a first illustration of our construction, we can obtain the generalized intelligent states for the standard harmonic oscillator. Indeed, using the equations (64) and (65) and setting $g(N) = N$, we have

$$|\Psi\rangle = c_0 \exp\left[\left(\frac{\lambda-1}{\lambda+1}\right)\frac{(a^+)^2}{2}\right] \exp\left[\left(\frac{2z}{\lambda+1}\right) a^+\right] |0\rangle \tag{66}$$

where $|0\rangle$ is the ground state for the harmonic oscillator.

4. Fock–Bargmann representation

It is well known that the Fock–Bargmann representation enables one to find simpler solutions for a number of problems, exploiting the theory of analytical entire functions.

In this part of our work, generalizing the pioneering work of Bargmann [31] for the usual harmonic oscillator, we will study the Fock–Bargmann representation of the dynamical algebra generated by annihilation and creation operators corresponding to an arbitrary quantum system. We recall that in the Fock–Bargmann representation for the harmonic oscillator, the creation operator a^+ is the multiplication by z, while the operator a^- is the differentiation with respect to z.

We define the Fock–Bargmann space as a space of functions S which are holomorphic on a ring D in the complex plane. The scalar product is written with an integral of the form

$$\langle f_1\,|f_2\rangle = \int \overline{f_1\,(z)} f_2\,(z)\;\mathrm{d}\mu\,(z) \tag{67}$$

where $\mathrm{d}\mu\,(z)$ is the measure defined above (see equation (41)). The Fock–Bargmann representation of the dynamical algebra $\{a^+, a^-, G\}$ is a representation on Fock–Bargmann space such that the annihilation and the creation operators admit eigenvectors generating S.

Let $|h\rangle$ be a state of the Hilbert space $\mathcal{H}$

$$|h\rangle = \sum_{n=0}^{\infty} h_n |\psi_n\rangle \qquad \sum_{n=0}^{\infty} |h_n|^2 < \infty. \tag{68}$$

Following the construction of [31], any state $|h\rangle$ of $\mathcal{H}$ in the Fock–Bargmann representation is represented by a function of the complex variable z (using the coherent states associated with an arbitrary quantum system)

$$h(z) = \langle \bar{z}, a\, |h\rangle = \sum_{n=0}^{\infty} \frac{z^n \mathrm{e}^{\mathrm{i}\alpha e_n}}{\sqrt{f(n)}} h_n \tag{69}$$

where the variable z belongs to the domain D of definition of the eigenvalues of a^- (annihilation operator). In particular, to the basis vectors $|\psi_n\rangle$ there correspond the monominals

$$\psi_n(z) \equiv \langle \bar{z}, a\, |\psi_n\rangle = \frac{z^n \mathrm{e}^{\mathrm{i}\alpha e_n}}{\sqrt{f(n)}}. \tag{70}$$

Using the equations (69) and (70), we can prove easily the following result.

In the Fock–Bargmann representation, we realize the annihilation operator a^- by

$$a^- = z^{-1} g\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right) \tag{71}$$

the creation operator a^+ by

$$a^+ = z \tag{72}$$

and the operator number by

$$N = z\frac{\mathrm{d}}{\mathrm{d}z}. \tag{73}$$

The Fock–Bargmann representation exists if we have a measure such that

$$\int |z, \alpha\rangle \langle z, \alpha| \, \mathrm{d}\mu(z) = I_{\mathcal{H}}. \tag{74}$$

The existence of the measure, which was discussed previously for an arbitrary quantum system, ensures that the scalar product takes the form (67).

We note that in the case where

$$g\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right) = z\frac{\mathrm{d}}{\mathrm{d}z} \qquad \text{i.e.} \quad g(N) = N \tag{75}$$

we recover the well known Fock–Bargmann representation of the harmonic oscillator.

An interesting case concerns the situation where

$$g(N) = N(N + \upsilon) \tag{76}$$

which occurs, for instance, when one deals with a quantum system evolving in the infinite square well or Pöschl–Teller potentials. The Fock–Bargmann realization presented in this section will be the corner stone to construct the generalized intelligent states for these potentials. This matter will be considered in section 5.

5. Pöschl–Teller intelligent states

We start by recalling the eigenvalues and eigenvectors of infinite square well and Pöschl–Teller potentials (cf section 5.1). We discuss the Gazeau–Klauder coherent states associated with these two quantum systems (cf section 5.2) and, using the Fock–Bargmann representation, we give an analytic realization of the generalized intelligent states corresponding to infinite square well and Pöschl–Teller potentials (cf section 5.3).

5.1. Spectrum of the Pöschl–Teller potentials

We consider the Hamiltonian

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{\kappa,\lambda}(x) \tag{77}$$

describing a particle on the line, and subjected to the potential

$$V_{\kappa,\lambda}(x) = \begin{cases} \dfrac{1}{4a^2}\left[\dfrac{\kappa(\kappa-1)}{\sin^2\left(\frac{x}{2a}\right)} + \dfrac{\lambda(\lambda-1)}{\cos^2\left(\frac{x}{2a}\right)}\right] - \dfrac{(\lambda+\kappa)^2}{4a^2} & 0 < x < \pi a \\ \infty & x \leqslant 0 \quad x \geqslant \pi a \end{cases} \tag{78}$$

for $\lambda > 1$ and $\kappa > 1$ (the parameter λ should not be confused with one appearing in the eigenvalue equation (15)). It is well known that the Pöschl–Teller potential [32] interpolates between the harmonic oscillator and the infinite square well. The infinite well takes place at the limit $\lambda = \kappa = 1$. The Hamiltonian H can be written in the following form:

$$H = a^+_{\kappa,\lambda} a^-_{\kappa,\lambda} \tag{79}$$

where the annihilation and creation operators are given by

$$a^{\pm}_{\kappa,\lambda} = \left(\mp\frac{\mathrm{d}}{\mathrm{d}x} + W_{\kappa,\lambda}(x)\right) \tag{80}$$

in terms of the superpotentials $W_{\kappa,\lambda}(x)$

$$W_{\kappa,\lambda}(x) = \frac{1}{2a}\left[\kappa \,\mathrm{cotg}\left(\frac{x}{2a}\right) - \lambda \tan\left(\frac{x}{2a}\right)\right]. \tag{81}$$

The eigenvectors are given by

$$\psi_n(x) = [c_n(\kappa,\lambda)]^{-\frac{1}{2}} \left(\cos\frac{x}{2a}\right)^{\lambda} \left(\sin\frac{x}{2a}\right)^{\kappa} \frac{n!\,\Gamma\left(\kappa+\frac{1}{2}\right)}{\Gamma\left(n+\kappa+\frac{1}{2}\right)} P_n^{\left(\kappa-\frac{1}{2},\lambda-\frac{1}{2}\right)}\left(\cos\frac{x}{a}\right) \tag{82}$$

where the normalization constant above is

$$c_n(\kappa,\lambda) = a\frac{\Gamma(n+1)\,\Gamma\left(\kappa+\frac{1}{2}\right)^2 \Gamma\left(n+\lambda+\frac{1}{2}\right)}{\Gamma\left(n+\kappa+\frac{1}{2}\right)\Gamma(n+\kappa+\lambda)\,\Gamma(2n+\kappa+\lambda)}. \tag{83}$$

The eigenvalues of H are given by

$$H|\psi_n\rangle = n(n+\kappa+\lambda)|\psi_n\rangle. \tag{84}$$

The creation and annihilation operators $a^+_{k,\lambda}$ and $a^-_{k,\lambda}$ act on $|\psi_n\rangle$ as follows:

$$\begin{aligned} a^+_{\kappa,\lambda}|\psi_n\rangle &= \sqrt{(n+1)(n+1+\kappa+\lambda)}\,\mathrm{e}^{-\mathrm{i}\alpha(2n+1+\kappa+\lambda)}|\psi_{n+1}\rangle \\ a^-_{\kappa,\lambda}|\psi_n\rangle &= \sqrt{n(n+\kappa+\lambda)}\,\mathrm{e}^{\mathrm{i}\alpha(2n-1+\kappa+\lambda)}|\psi_{n-1}\rangle. \end{aligned} \tag{85}$$

From the latter equation, one can verify that $a^+_{k,\lambda}$ and $a^-_{k,\lambda}$ satisfy the following commutation relations:

$$\left[a^-_{\kappa,\lambda}, a^+_{\kappa,\lambda}\right] = G_{\kappa,\lambda}(N) \tag{86}$$

where

$$G_{\kappa,\lambda}(N) \equiv G = 2N + (1+\kappa+\lambda) \tag{87}$$

and the operator N is defined, as in the first section, by

$$N|\psi_n\rangle = n|\psi_n\rangle. \tag{88}$$

Here also, we mention that $N \neq a^+_{\kappa,\lambda}a^-_{\kappa,\lambda} = H$.

5.2. Coherent states for the Pöschl–Teller potentials

From the result obtained in section 3, the coherent states read as

$$|z,\alpha\rangle = \mathcal{N}(|z|)\sum_{n=0}^{\infty}\frac{z^n \mathrm{e}^{-\mathrm{i}\alpha n(n+\kappa+\lambda)}}{\sqrt{\Gamma(n+1)\Gamma(n+\kappa+\lambda+1)}}\,|\psi_n\rangle\,. \tag{89}$$

The normalization constant is given by

$$\mathcal{N}(|z|) = \sqrt{\frac{|z|^{\kappa+\lambda}}{I_{\kappa+\lambda}(2|z|)}}. \tag{90}$$

It is easy to verify that the radius of convergence $\mathcal{R}$ is infinite.

The identity resolution is explicitly given by

$$\int |z,\alpha\rangle\langle z,\alpha|\,\mathrm{d}\mu(z) = I_{\mathcal{H}} \tag{91}$$

where the measure $\mathrm{d}\mu(z)$ can be computed by the inverse Mellin transform [33]

$$\mathrm{d}\mu(z) = \frac{2}{\pi}I_{\kappa+\lambda}(2r)\,K_{\frac{\kappa+\lambda}{2}}(2r)\,r\,\mathrm{d}r\,\mathrm{d}\phi \qquad z = r\mathrm{e}^{\mathrm{i}\phi}. \tag{92}$$

The coherent states of the infinite square well are obtained from the Pöschl–Teller ones simply by putting $\lambda+\kappa = 2$.

The coherent states from an overcomplete family of states (resolving the identity resolution by integration with respect to the measure given by (92)), and provide a representation of any state $|\Psi\rangle$ by an entire analytic function $\langle\Psi\,|z,\alpha\rangle$.

Using the Fock–Bargmann representation discussed above, the creation and annihilation operators, corresponding to the quantum system evolving in Pöschl–Teller (or in the infinite square well) potentials, are realized by

$$a^{+}_{\kappa,\lambda} = z \qquad a^{-}_{\kappa,\lambda} = z\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (\lambda+\kappa+1)\frac{\mathrm{d}}{\mathrm{d}z}. \tag{93}$$

It is easy to see that the operator G, in this representation, acts as

$$G = 2z\frac{\mathrm{d}}{\mathrm{d}z} + (\lambda+\kappa+1)\,. \tag{94}$$

We will use this representation to construct Pöschl–Teller generalized intelligent states. Those corresponding to the infinite square well can be obtained simply by taking $\lambda = \kappa = 1$.

5.3. Pöschl–Teller generalized intelligent states

We consider the Pöschl–Teller equal variance $|z,\alpha\rangle$ (the eigenstates of the annihilation operator $a^{-}_{\kappa,\lambda}$). As we mentioned before, these states provide a representation of any state $|\Phi\rangle$ (belonging to the Hilbert space corresponding to the system evolving in the Pöschl–Teller potential) by an entire analytical function $\langle\Phi\,|z,\alpha\rangle$. Then, by means of the analytical realization and using the differential representation of the creation and annihilation operators, we can construct the Pöschl–Teller generalized intelligent states $\left|z',\lambda,\alpha\right\rangle$ (we denote for a while the eigenvalue by z' and we put $\lambda+\kappa = \upsilon$).

The eigenvalue equation (23) which takes the form

$$\left[(1+\lambda)\,a^{-} + (1-\lambda)\,a^{+}\right]\left|z',\lambda,\alpha\right\rangle = 2z'\left|z',\lambda,\alpha\right\rangle \tag{95}$$

now reads

$$\left[(1+\lambda)\left(z\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (\upsilon+1)\frac{\mathrm{d}}{\mathrm{d}z}\right) + (1-\lambda)\,z\right]\Phi_{(z',\lambda,\alpha)}(z) = 2z'\Phi_{(z',\lambda,\alpha)}(z)\,. \tag{96}$$

By means of a simple substitution, the above equation is reduced to the Kummer equation for the confluent hypergeometric function ${}_1F_1\,(a, b, z)$, so we have the following solution of equation (96):

$$\Phi_{(z',\lambda,\alpha)}\,(z) = \exp\left(\sqrt{\frac{\lambda-1}{\lambda+1}}z\right){}_1F_1\left(a, b, -2\sqrt{\frac{\lambda-1}{\lambda+1}}z\right) \tag{97}$$

where

$$a = \frac{\upsilon+1}{2} - \frac{z'}{\sqrt{(\lambda^2-1)}} \qquad \text{and} \qquad b = \upsilon + 1. \tag{98}$$

Using the properties of the above hypergeometric function (cf equation (97)), we arrive at the conclusion that the squeezing parameter λ obeys the following condition:

$$\sqrt{\left|\frac{1-\lambda}{1+\lambda}\right|} < 1 \qquad \Longleftrightarrow \qquad \operatorname{Re}\lambda > 0 \tag{99}$$

which is exactly the restriction on λ imposed by the positivity of the commutator $\left[a^-, a^+\right] = G\,(N)$ (see equation (18)). Thus we obtain the Pöschl–Teller generalized intelligent states in the coherent state representation in the form (up to the normalization constant)

$$\left\langle z', \lambda, \alpha\right|\left. z, \alpha\right\rangle = \exp\left(c^* z\right){}_1F_1\left(a^*, b, -2c^* z\right) \tag{100}$$

where

$$c = \sqrt{\frac{\lambda-1}{\lambda+1}} \tag{101}$$

and the parameters a and b are defined in formulae (98). In the case where $\lambda = 1$ (i.e. $c = 0$), using the power series of ${}_1F_1\,(\alpha, \beta, x)$

$${}_1F_1\,(\alpha, \beta, x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n}\frac{x^n}{n!} \qquad \text{where} \quad (\alpha)_n = \alpha\,(\alpha+1)\cdots(\alpha+n-1) \tag{102}$$

we obtain

$$\left\langle z', \lambda = 1, \alpha\right|\left. z, \alpha\right\rangle = {}_0F_1\left(\upsilon + 1, z\overline{z'}\right) \tag{103}$$

where

$${}_0F_1\,(\alpha, x) = \sum_{n=0}^{\infty} \frac{1}{(\alpha)_n}\frac{x^n}{n!}. \tag{104}$$

The result (103) coincides with the solution (89) for $\lambda = 1$, and we recover the Pöschl–Teller coherent states defined as the $a^-_{\kappa,\lambda}$ eigenvectors.

To close this section, we discuss the coherence and squeezing of Pöschl–Teller generalized intelligent states.

As discussed above, in the case where $\lambda = 1$, we have the so-called Gazeau–Klauder coherent states. The dispersions $\Delta W \equiv \Delta W_{\kappa,\lambda}$ and ΔP saturate the Robertson–Schrödinger uncertainty relation and we obtain

$$(\Delta W)^2 = (\Delta P)^2 = \tfrac{1}{2}\,\langle G\rangle \tag{105}$$

and $\langle F\rangle = 0$. Using the expression of the operator G in the case of Pöschl–Teller (or infinite square well) potentials, one can show that its mean value on the coherent states $|z, \alpha\rangle$ is given by

$$\langle G\rangle = \langle z, \alpha|\,G(N)\,|z, \alpha\rangle = (1+\upsilon) + \frac{2\,|z|^2}{(1+\upsilon)}\frac{{}_0F_1\left(2+\upsilon, |z|^2\right)}{{}_0F_1\left(1+\upsilon, |z|^2\right)}. \tag{106}$$

Note also that

$$\langle G\rangle \geqslant 1 + \upsilon \geqslant \tfrac{1}{2} \tag{107}$$

which traduces the fact that the dispersions ΔP and ΔW are greater than $\frac{1}{2}$ (remember that in the case of the harmonic oscillator we have $\Delta P = \Delta W = \frac{1}{2}$). This result constitutes a main difference from the well known harmonic oscillator coherent states.

Another interesting situation concerns the case $|\lambda| = 1$ with $\lambda \neq \pm 1$. The case $\lambda = 1$ was discussed before and $\lambda = -1$ is not allowed by our construction. Taking $\lambda = \mathrm{e}^{\mathrm{i}\theta}$ ($\theta \neq k\pi$; $k \in \mathbb{N}$), the states are coherent and dispersions are given by

$$(\Delta W)^2 = (\Delta P)^2 = \frac{1}{2\,|\cos\theta|}\,\langle G\rangle\,. \tag{108}$$

The mean value of the operator F is non-vanishing (vanishing only in the Gazeau–Klauder coherent states). It is given by

$$\langle F\rangle = \mathrm{tg}\theta\,\langle G\rangle\,. \tag{109}$$

From the latter equation, we conclude that the presence of the correlation does not forbid the system being prepared in a coherent state. This result is true for any quantum system. The properties of generalized intelligent states turn out to be sensitive to the spectral properties of the commutator $\left[a^-_{\kappa,\lambda}, a^+_{\kappa,\lambda}\right] = G(N)$.

Consider now (for the sake of completeness) the exceptional case of states which minimize the Robertson–Schrödinger uncertainty relation with $\mathrm{Re}\,\lambda = 0$. In this case, we have eigenstates with vanishing mean value of G. In the same way, the mean value of F, on the generalized intelligent states with $\mathrm{Im}\,\lambda = 0$, is zero. Finally, in the case where $|\lambda| \neq 1$, the generalized intelligent states exhibit strong squeezing. This takes place, for example, when $\lambda \to 0$ (cf equation (16)), which can be easily derived even without explicit calculation of the variances.

6. Conclusion

In this paper, we gave a complete classification of eigenstates of the eigenvalue equation arising from the minimization of the Robertson–Schrödinger uncertainty relation. We obtained the so-called generalized squeezed states for $|\lambda| \neq 1$ and the generalized coherent states for $|\lambda| = 1$. The latter class includes the Gazeau–Klauder coherent states for which we examined the properties known for them such as continuity, temporal stability, action identity and resolution of unity. The measure, which ensures the overcompleteness of coherent states, is strongly related to the nature of the spectrum under study. We also purposed a Fock–Bargmann representation of the creation and annihilation operators for an arbitrary quantum system. This representation was useful to construct, in an analytical way, the generalized intelligent states for the Pöschl–Teller and infinite square well potentials. The results obtained through this work constitute a first step to obtain more information about the squeezing and coherence for an arbitrary quantum system and we believe that there are many directions on this subject which can be explored. Indeed, we think that our results can be adapted to the x^4 anharmonic oscillator [34]. We also hope to construct the Perelomov coherent state types (group theoretical approach) for an arbitrary quantum system and compare them with Gazeau–Klauder ones. These matters will be considered in a forthcoming work.

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